How to get Rapid Thermalization from Perturbative QCD in Heavy Ion Collisions

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1 – Introduction

Thermalization

- An old question but one that still haunts us from the very beginning!
- Tough question to answer
  $\Longrightarrow$ Take the easy way out!
  $\Longrightarrow$ Assume fast thermalization!
- Parton based numerical models:
  PCM, work by S.W., others ... etc.
  The system clearly approaches towards equilibrium but *not nearly as fast as* one would have liked.
- Recently from RHIC data:
  Elliptic flow $v_2(p_t)$ measurements for π and $p + \bar{p}$ agree with results from hydrodynamics model!!!
- But numerical models says:
  Hydrodynamics shouldn't work!
- Then RHIC says:
  If hydro works, then kinetic equilibration has to be very, very fast indeed!!
- HOW??
  Something non-perturbative or perturbative?

2 – Some Basic Facts

In the initial parton phase, we know:

- Small Angle Scattering
  - large amplitude
  - inefficient for momentum rearrangement
  - forward-backward cancellation
- Large Angle Scattering
  - small amplitude
  - efficient for momentum rearrangement
  - much less forward-backward cancellation
- Elastic Scattering
  - by no means the most important for thermalization
  - insufficient for thermalization
  - need inelastic processes
  - must go beyond lowest order in α_s
- How far beyond?
- From past experience
  $2 \leftrightarrow 2 \sim 2 \leftrightarrow 3$ are of comparable size. Including both is still not enough!
- Need to go much further beyond!
- Need simplification!

3 – Simplification

- Parton plasma is gluon dominated
  $\Longrightarrow$ only consider gluons!
- Large angle scattering is efficient for thermalization but is small in magnitude!
- Small angle scattering is not efficient but is large!

Solution: Large Angle + Small Angle

For example considering only $gg \leftrightarrow (n-2)g$ processes

The aims

- Efficient momentum rearrangement, and
- Sizeable amplitudes

  $\Longrightarrow$ Largest collision terms

The means

- One and only one large angle scattering $\forall$ values of n.
- The rest of the gluons are
  emitted or absorbed nearly collinearly.
- Hopefully we can get a boost in the large angle!

7 – Summary/Outlook/Remarks

- Use

Large Angle Hard Collisions

$\Uparrow\Uparrow\Uparrow\Uparrow$

Small Angle Emissions-Absorptions

To boost thermalization!

- Multi-gluon processes $mg \leftrightarrow (n-m)g$:
  - Sum up collinear gluons.
  - Arrange into $\sim$ a factor $\times$ $(2 \leftrightarrow 2)$.
- Enhancement is Q^2 dependent!
- Early on, most collisions are hard
  $\Longrightarrow$ large to fairly large Q^2 enables fast momentum rearrangement!
- At RHIC, usually quoted initial temperature:
  $T \sim 500$ MeV
  - when it does not typically give very large Q^2,
  - if this mechanism is correct, kinetic equilibrium must have been completed by then,
  - it seems to be consistent with data so far.
- Not our last words, still on-going investigation!

Impose one large angle scattering + small angle emission-absorption for the rest.

E.g. gluon $4\|5$ in $C_{12\leftrightarrow 345}(p_1)$

In the collinear limit, relabel $(4,5) \to (i,j)$

$$\sum |\mathcal{M}_{12\to 3ij}|^2 \simeq \frac{8\pi\alpha_s(s_{ij})}{s_{ij}} P_{gg}(z) \sum |\mathcal{M}_{12\to 34}|^2$$

where

$$P_{gg}(z) = 2\,N_c \left(\frac{1-z}{z} + \frac{z}{1-z} + z(1-z) \right) .$$

$$\int [dp_i][dp_j] D_{12,3ij} = \int [dp_{4^*}] D_{12,34^*} \int \frac{ds_{ij}}{2\pi} [dp_i][dp_j] D_{4^*,ij}$$

$$\int \frac{ds_{ij}}{2\pi} [dp_i][dp_j] D_{4^*,ij} = \int \frac{dz\; ds_{ij}}{16\pi^2}$$

$$\int d\Phi_{3ij} D_{12,3ij} \sum |\mathcal{M}_{12\to 3ij}|^2$$

$$\simeq \int d\Phi_{34} D_{12,34} \sum |\mathcal{M}_{12\to 34}|^2 \int \frac{dz\; ds_{ij}}{s_{ij}} \frac{\alpha_s(s_{ij})}{2\pi} P_{gg}(z) .$$

Large angle part

Small angle part

Therefore for time-like splitting and coalescence

$$C^{4\|5}_{12\leftrightarrow 345}(p) \simeq -\frac{1}{\nu_g}\int [dp_2]d\Phi_{34}D_{12,34}\frac{1}{3!}\sum|\mathcal{M}_{12\to 34}|^2$$
$$\times \int dz \frac{ds_{45}}{s_{45}}\frac{\alpha_s(s_{45})}{2\pi}P_{gg}(z)$$
$$\times\Big[f_1 f_2(1+f_3)(1+f_4(z))(1+f_4(1-z))$$
$$-(1+f_1)(1+f_2)f_3 f_4(z)f_4(1-z)\Big]$$

where $f_i(z) = f(zp_i)$.

Similarly for gluon $2\|5$ for space-like splitting and absorption, go through similar steps to get

$$C^{2\|5}_{12\leftrightarrow 345}(p_1) \simeq -\frac{1}{\nu_g}\int [dp_2]d\Phi_{34}D_{12,34}\frac{1}{3!}\sum|\mathcal{M}_{12\to 34}|^2$$
$$\times \int dx \frac{dt_{25}}{t_{25}}\frac{\alpha_s(|t_{25}|)}{2\pi}P_{gg}(x)$$
$$\times\Big[f_1 f_2(\tfrac{1}{x})(1+f_3)(1+f_4)(1+f_2(\tfrac{1-x}{x}))$$
$$-(1+f_1)(1+f_2(\tfrac{1}{x}))f_3 f_4 f_2(\tfrac{1-x}{x})\Big].$$

Convolution of

hard binary collision with small angle subprocess!

5.2 – Virtual Corrections

In the vacuum, according to Altarelli-Parisi the probability distribution

$$\mathcal{P}_{gg} + d\mathcal{P}_{gg} = \delta(1-z) + \frac{\alpha_s}{2\pi} P^{+}_{gg}(z) ds$$

where $P^{+}_{gg}(z)$ is the regularized gluon splitting function or the kernel of the pure glue part of DGLAP evolution equation.

Momentum conservation (in the absence of $q\bar{q}$)

$$\Longrightarrow \int_0^1 dz z P^{+}_{gg}(z) = 0$$

or writing

$$P^{+}_{gg}(z) = P_{gg}(z)|_{+} + \mathcal{C}_{T=0}\delta(1-z) \ ,$$

where $\mathcal{C}_{T=0}$ is a constant and $|_{+}$ denotes plus-distribution prescription, which implies

$$\int_0^1 dz \Big\{ z P_{gg}(z)|_{+} + \mathcal{C}_{T=0}\, \delta(1-z) \Big\} = 0 \ .$$

It follows

$$\mathcal{C}_{T=0} = -\int_0^1 dz z P_{gg}(z)|_{+} \ .$$

In a QCD medium, $\mathcal{P}_{gg}$ for $g^* \to gg$ is now

$$
\begin{aligned}
&\mathcal{P}_{gg} + d\mathcal{P}_{gg} \\
&= \delta(1-z) + \frac{\alpha_s}{2\pi}\Bigg(C^{\text{emit}}_{T\,\text{out}}(p)\delta(1-z) \\
&\qquad + P_{gg}(z)|_+ \frac{\big(1+f(zp)\big)\big(1+f((1-z)p)\big)}{1+f(p)} \Bigg) ds
\end{aligned}
$$

Momentum conservation

$$\int_0^1 dz z \Big(\ldots\ldots \Big) = 0 \,.$$

$$C^{\text{emit}}_{T\,\text{out}}(p) = -\int_0^1 dz z P_{gg}(z) \frac{\big(1+f(zp)\big)\big(1+f((1-z)p)\big)}{1+f(p)}$$

Adding virtual corrections $\Longrightarrow 1-z$ factor!

For other $gg \to g^*$, $g \to g^* g$, and $g^* g \to g$ there are similar quantities $C^{\text{abs}}_{T\,\text{in}}(p)$, $C^{\text{emit}}_{T\,\text{in}}(p)$, and $C^{\text{abs}}_{T\,\text{out}}(p)$, respectively. They differ from $C^{\text{emit}}_{T\,\text{out}}(p)$ only in the product of distributions.

Now including virtual corrections,

our example: for gluon $4\|5$

$$C^{4\|5}_{12\leftrightarrow 345}(p) \simeq -\frac{1}{\nu_g}\int [dp_2] d\Phi_{34} D_{12,34} \frac{1}{3!}\sum |\mathcal{M}_{12\to 34}|^2$$

$$\times \int dz \frac{ds_{45}}{s_{45}} \frac{\alpha_s(s_{45})}{2\pi}(1-z)P_{gg}(z)$$

$$\times \Big[f_1 f_2 (1+f_3)(1+f_4(z))(1+f_4(1-z))$$

$$-(1+f_1)(1+f_2) f_3 f_4(z) f_4(1-z)\Big] .$$

Manipulate expression to resemble the binary collision:

$$C^{4\|5}_{12\leftrightarrow 345}(p) \simeq -\frac{1}{\nu_g}\int [dp_2] d\Phi_{34} D_{12,34} \frac{1}{3!}\sum |\mathcal{M}_{12\to 34}|^2$$

$$\times \Big[f_1 f_2 (1+f_3)(1+f_4) F^{\mathrm{emit}}_{\mathrm{out}}(p_4, Q^2)$$

$$-(1+f_1)(1+f_2) f_3 f_4 F^{\mathrm{abs}}_{\mathrm{in}}(p_4, Q^2)\Big] .$$

where Q^2 is the momentum transfer of $12 \to 34$ and

$$F^{\mathrm{emit}}_{\mathrm{out}}(p,s) = \int \frac{dz\, ds_{ij}}{s_{ij}} \frac{\alpha_s(s_{ij})}{2\pi}(1-z)P_{gg}(z)$$

$$\times \Big(1+f((1-z)p)\Big)\Big(1+f(zp)\Big)/\Big(1+f(p)\Big)$$

$$F^{\mathrm{abs}}_{\mathrm{in}}(p,s) = \int \frac{dz\, ds_{ij}}{s_{ij}} \frac{\alpha_s(s_{ij})}{2\pi}(1-z)P_{gg}(z)$$

$$\times f((1-z)p) f(zp)/f(p)$$

5.3 – $C_4(p_1) + C_5(p_1)$ in the Collinear Limit

For simplicity, let

- gluon 1 with momentum p_1 be one of the four main gluons
- no collinear emission from or absorption by gluon 1

Otherwise collecting all possibilities of collinear emission and absorption!

$$
\begin{aligned}
& C_{12\leftrightarrow 34}(p_1) + C^{\|}_{12\leftrightarrow 345}(p_1) + C^{\|}_{123\leftrightarrow 45}(p_1) \\
\simeq & -\frac{1}{\nu_g}\int [dp_2] d\Phi_{34} D_{12,34} \frac{1}{2!}\sum |\mathcal{M}_{12\to 34}|^2 \\
& \times \Big[f_1 f_2 (1+f_3)(1+f_4) \\
& \quad \times \Big(1 + F^{\mathrm{abs}}_{\mathrm{in}}(p_2, Q^2) + F^{\mathrm{emit}}_{\mathrm{out}}(p_3, Q^2) + F^{\mathrm{emit}}_{\mathrm{out}}(p_4, Q^2) \\
& \qquad + F^{\mathrm{emit}}_{\mathrm{in}}(p_2, Q^2) + F^{\mathrm{abs}}_{\mathrm{out}}(p_3, Q^2) + F^{\mathrm{abs}}_{\mathrm{out}}(p_4, Q^2) \Big) \\
& \quad -(1+f_1)(1+f_2) f_3 f_4 \\
& \quad \times \Big(1 + F^{\mathrm{emit}}_{\mathrm{out}}(p_2, Q^2) + F^{\mathrm{abs}}_{\mathrm{in}}(p_3, Q^2) + F^{\mathrm{abs}}_{\mathrm{in}}(p_4, Q^2) \\
& \qquad + F^{\mathrm{abs}}_{\mathrm{out}}(p_2, Q^2) + F^{\mathrm{emit}}_{\mathrm{in}}(p_3, Q^2) + F^{\mathrm{emit}}_{\mathrm{in}}(p_4, Q^2) \Big) \Big] .
\end{aligned}
$$

$F(p, Q^2) > 0$ Enhance the binary large angle collision!

6 – Beyond the 5-gluon Processes

(Still on-going work)

Preliminary remarks: In general,

- Multi-gluon processes with one large angle collision each
- Many collinear gluon emissions and absorptions:
  - Fairly large Log's to compensate for small α_s!
  - Have to sum up these Log's
- Schematically, can arrange

$$C(p_1) = C_4 + \sum_{n=5} C_n(p_1)$$

  In the collinear limit, very roughly

$$C^{\parallel}(p_1) \sim C_4(p_1) \otimes \mathbb{F}(p_1, Q^2)\mathbb{F}(p_2, Q^2)\mathbb{F}(p_3, Q^2)\mathbb{F}(p_4, Q^2)$$

- $\mathbb{F}(p, Q^2)$'s satisfy non-linear, coupled differential equations
  - Have to be solved numerically!
- Try linearization to get uncoupled equations
  - Only partial enhancement is included!
  - Still get sizable enhancement factors!

4 – Relative Sizes of the $n+1$ to n Process

Make rough estimate, using

$$|\mathcal{M}_{gg\leftrightarrow(n-2)g}|^2 \sim F_{KS}(n)\, |\mathcal{M}_n^{\rm PT}|^2$$

where

$$|\mathcal{M}_n^{\rm PT}|^2 = \frac{g_s^{2n-4} N_c^{n-1}}{N_c^2-1} \sum_{i>j} s_{ij}^4 \sum_{perm.} \frac{1}{s_{12}s_{23}\cdots s_{n1}}$$

is the Parke-Taylor formula, $s_{ij} = (p_i + p_j)^2$ and

$$F_{KS}(n) = \frac{2^n - 2(n+1)}{n(n-1)}$$

is the Kunszt-Stirling factor.

- 4 gluons define the single large angle scattering scattering into deficient region in $\boldsymbol{p}$-space
- The other (n-4) gluons collinear to these!

$$s_{ij} \begin{array}{l} \nearrow \; s_l \qquad\qquad\qquad\qquad \text{— large} \\ \searrow \; s_s \sim m_t^2 = m_D^2/2 \text{ — small} \end{array}$$

- Count s_l^4 terms in the numerator
- Count terms with largest no. of s_s^4 in the denominator

One gets

$$|\mathcal{M}_n^{\rm PT}|^2 \simeq \frac{g_s^{2n-4} N_c^{n-1}}{N_c^2-1}\, 3(n-2) s_l^4\, \frac{3!(n-3)!}{2\, s_l^4 s_s^{n-4}}$$

therefore

$$\frac{|\mathcal{M}_{(n+1)}^{\rm PT}|^2}{|\mathcal{M}_n^{\rm PT}|^2} \simeq \frac{4\pi\alpha_s N_c (n-1)}{s_s}\,.$$

Ratio of $(n+1)$ to n collision terms

$$R(n) \simeq \frac{F_{KS}(n+1)}{F_{KS}(n)} \frac{4\pi\alpha_s N_c}{m_t^2} \frac{(n-1)}{(n-2)} \frac{m_D^2}{48\pi\alpha_s}$$

n	$R(n)$	$\prod_{i=4}^{n} R(i)$
4	1.50	1.50
5	1.11	1.67
6	1.00	1.67
7	0.96	1.59
8	0.94	1.49
10	0.93	1.29
20	0.96	0.72
30	0.97	0.50
100	0.99	0.16

Higher terms are of sizeable importance!!

5 – Perturbative Approach

Transport equation

$$p^\mu \frac{\partial f(p)}{\partial x^\mu} = C(p)$$

5.1 – 5-gluon processes

$$C_{12\leftrightarrow 345}(p_1) = -\frac{1}{\nu_g}\int [dp_2] d\Phi_{345} D_{12,345} \frac{1}{3!}\sum |\mathcal{M}_{12\to 345}|^2$$
$$\times\Big[f_1 f_2 (1+f_3)(1+f_4)(1+f_5)$$
$$-(1+f_1)(1+f_2) f_3 f_4 f_5\Big]$$

$$C_{123\leftrightarrow 45}(p_1) = -\frac{1}{\nu_g}\int d\Phi_{23} d\Phi_{45} D_{123,45} \frac{1}{2!2!}\sum |\mathcal{M}_{123\to 45}|^2$$
$$\times\Big[f_1 f_2 f_3 (1+f_4)(1+f_5)$$
$$-(1+f_1)(1+f_2)(1+f_3) f_4 f_5\Big].$$

where $\nu_g = 2\times 8$,
$f_i = f(p_i)$, $D_{12,345} = (2\pi)^4\delta^4(p_1+p_2-p_3-p_4-p_5)$,
$[dp] = \frac{d^3\boldsymbol{p}}{(2\pi)^3 2p^0}$, $\int d\Phi_{12\ldots m} = [dp_1][dp_2]\ldots[dp_m]$.